Supplementary Appendix for "Location Sorting and Endogenous Amenities: Evidence from Amsterdam"

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S.1 Proof of existence of equilibrium

A stationary equilibrium is guaranteed to exist under mild conditions. To see this, define the excess demand for short-term housing as follows,

$$z_j^S(\mathbf{r}, \mathbf{p}, \mathbf{a}) \equiv \mathcal{Q}_j^S(\mathbf{p}, \mathbf{a}) - \mathcal{H}_j^S(r_j, p_j; \kappa_j) \quad \forall j \in \mathcal{J}.$$

Similarly, define the excess demand for short-term housing as follows,

$$z_j^L(\mathbf{r}, \mathbf{p}, \mathbf{a}) \equiv \mathcal{Q}_j^{D,L}(\mathbf{r}, \mathbf{a}) - \mathcal{H}_j^L(r_j, p_j; \kappa_j) \quad \forall j \in \mathcal{J}.$$

Assumption 1 *We assume that each type of households has a minimum area of housing necessary for subsistence; let us denote this level by $\gamma_k, \forall k$. Denote by $\psi_k \equiv w_k / \gamma_k$ the maximum rental price that a household of type k is able to pay to reach its subsistence level of housing. We define*

$$\psi_{min} \equiv \min_k \psi_k \quad \textit{and} \quad \psi_{max} \equiv \max_k \psi_k$$

Further, we assume there is a constant $q \in (0, 1]$ such that

$$q \times \sum_{k=1}^{K} M_k \cdot \gamma_k = \sum_{j=1}^{J} H_j$$

In words, this assumption says that the city can accommodate at most fraction q of households.

Proposition 1 *Suppose Assumption 1 holds. Then a stationary equilibrium exists in the dynamic model.*

Proof. We perform the change of variables to $\tilde{r}_j \equiv \log(r_j)$ and seek a vector of

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log-rent, a vector of Airbnb prices, and a matrix of amenities that clear all markets.

In what follows, define

$$\mathcal{D} \equiv \bigtimes_j \left[\min\left\{ -H_j, \log\left(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\right)\right\}, \log\left(\max\left\{ \sum_k \frac{\alpha_h^k w^k M_k}{\psi_{min}}, \psi_{max} \right\}\right)\right] \\ \times \left[0, \frac{1}{F_{js}\sigma_s}\Big(\sum_{k=1}^K M^k \alpha_s^k \alpha_c^k w^k + M^T \alpha_s^T \alpha_c^T w^T \Big)\right]^{J \cdot S}$$

and observe $\mathcal{D}$ is convex and compact. The choice of the domain for the household problem will become clear later in the proof. We also denote by ω a generic concatenation of $\widetilde{\mathbf{r}}$ and $\mathbf{a}$. We can drop short-term rental prices for now as they do not appear in the decision problem of household.

First, we prove that the value function of households is well-defined and unique. Recall that the value function for household i of type k is

$$V^k(x_{it}, \omega_t, \varepsilon_{it}) = \max_d \left\{ u^k(j, x_{it}, \omega_t) + \epsilon_{idt} + \beta \mathbb{E}\left[V^k(x_{it+1}, \omega_{t+1}, \varepsilon_{it+1}) | d, x_{it}, \omega_t, \varepsilon_{it} \right] \right\}$$

Then, we have

$$\begin{aligned} EV^k(x_{it}, \omega_t) &\equiv \mathbb{E}_{\omega_t, \varepsilon_{it}}\left[V^k(x_{it}, \omega_t, \varepsilon_{it}) | x_{it-1}, \omega_{t-1}, \varepsilon_{it-1} \right] \\ &= \mathbb{E}_{\omega_t, \varepsilon_{it}}\left[\max_d \left\{ u^k(j, x_{it}, \omega_t) + \varepsilon_{idt} + \beta EV^k(x_{it+1}, \omega_{t+1}) \right\} | x_{it}, \omega_t, \varepsilon_{it} \right] \\ &= \mathbb{E}_{\omega_t | x_{it-1}, \omega_{t-1}} \mathbb{E}_{\varepsilon_{it} | \omega_t, x_{it}, \omega_{t-1}}\left[\max_d \left\{ u^k(j, x_{it}, \omega_t) + \varepsilon_{idt} + \beta EV^k(x_{it+1}, \omega_{t+1}) \right\} \right] \\ &= \mathbb{E}_{\omega_t | x_{it}, \omega_{t-1}}\left[\log\Big(\sum_d \exp\left(u^k(j, x_{it}, \omega_t) + \beta EV^k(x_{it+1}, \omega_{t+1}) \right) \Big) \right] \\ &= \int_{\omega_t} \log\Big(\sum_d \exp\left(u^k(j, x_{it}, \omega_t) + \beta EV^k(\iota(j, x_{it+1}), \omega_t) \right) \Big) p(d\omega_t | x_{it}, \omega_t - 1) \end{aligned}$$

where $p(d\omega' | x, \omega)$ is a transition function for the exogenous state variables. We used the fact that the ε's are distributed i.i.d. mean-zero Type I EV. Moreover, since we seek a stationary equilibrium and assume households have perfect foresight, the notation above can be compressed to

$$EV^k(x_{it}, \omega_t) = \log\Big(\sum_d \exp\big(u^k(j, x_{it}, \omega_t) + \beta EV^k(x_{it+1}, \omega_{t+1})\big)\Big)$$

Let $\mathcal{B}(\mathcal{D})$ be the Banach space of continuous and bounded functions on the compact set $\mathcal{D}$ defined above. Define an operator $T^k : \mathcal{B}(\mathcal{D}) \to \mathcal{B}(\mathcal{D})$ by

$$T^k(EV^k)(x_{it}, \omega_t) = \log\Big(\sum_d \exp\big(u^k(j, x_{it}, \omega_t) + \beta EV^k(x_{it+1}, \omega_{t+1})\big)\Big)$$

This is well-defined. Indeed, if EV^k is continuous, $T^k(EV^k)$ is continuous. Suppose EV^k is bounded. Since the domain for amenities and prices is compact and the flow utility, exponential and logarithmic functions are continuous, we have by Weierstrass' theorem that $T^k(EV^k)$ is also bounded, and hence belongs to $\mathcal{B}(\mathcal{D})$. Therefore, T is well-defined.

Next, we show that T is a contraction mapping on $\mathcal{B}(\mathcal{D})$. To this end, note that $\forall (x, \omega) \in \mathcal{D}$ and $g, h \in \mathcal{B}(\mathcal{D})$, we have

$$\max_{d \in D}\Big\{u^k(j, x, \omega) + \varepsilon_d + \beta g(x', \omega')\Big\} - \max_{d \in D}\Big\{u^k(j, x, \omega) + \varepsilon_d + \beta h(x', \omega')\Big\} \leqslant \max_{d \in D} |g(x', \omega') - h(x', \omega')|$$

Hence, taking sup over $\mathcal{D}$, it follows that

$$||T(g) - T(h)||_\infty \leqslant \beta ||g - h||_\infty$$

This shows that T is a contraction mapping. Hence, EV^k is the unique fixed point of this operator, and is a continuous and bounded function.

We also claim that EV^k is a strictly decreasing function in $\tilde{r}_j, \forall j \in \mathcal{J}$ with $\tilde{r}_j < \log(\psi_k)$ and decreasing otherwise. To this end, first note that the set of decreasing continuous and bounded functions over $\mathcal{D}$, which we denote by $\mathcal{B}_D(\mathcal{D}) \subseteq \mathcal{B}(\mathcal{D})$, is a closed subset of $\mathcal{B}(\mathcal{D})$, and hence a Banach space itself. Since u^k is a decreasing function of $\tilde{r}_j, \forall j \in \mathcal{J}$, it follows that the Bellman operator T^k maps $T^k(\mathcal{B}_D(\mathcal{D})) \subseteq \mathcal{B}_D(\mathcal{D})$. By a well-known corollary to the Contraction Mapping Theorem (see Corollary 1 to Theorem 3.2. in Stokey, Lucas and Prescott (1989)), it follows $EV^k \in \mathcal{B}_D(\mathcal{D})$. Further, restrict $\mathcal{D}$ to $\widetilde{\mathcal{D}}$ which we define to be

$$\widetilde{\mathcal{D}} \equiv \left\{ (\mathbf{r}, \mathbf{a}) \in \mathcal{D} : \tilde{r}_j \in \left[\min\left\{ -H_j, \log\left(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j} \right) \right\}, \log(\psi_k) \right], \forall j \in \mathcal{J} \right\}$$

Note that $\widetilde{\mathcal{D}}$ is a closed subset of $\mathcal{D}$, and, hence, compact. Then, $\mathcal{B}(\widetilde{\mathcal{D}})$ and $\mathcal{B}_D(\widetilde{\mathcal{D}})$ are Banach spaces. Let $\mathcal{B}_{SD}(\widetilde{\mathcal{D}})$ be the set of strictly decreasing continuous and bounded functions on $\widetilde{\mathcal{D}}$. Observe that $T^k(\mathcal{B}_D(\widetilde{\mathcal{D}})) \subseteq \mathcal{B}_{SD}(\widetilde{\mathcal{D}})$. It follows by a corollary to the Contraction Mapping Theorem (Corollary 1 to Theorem 3.2. in Stokey et al. (1989)) that $EV^k \restriction_{\widetilde{\mathcal{D}}} \in \mathcal{B}_{SD}(\widetilde{\mathcal{D}})$.

Notice that for an action d, x_{it+1} is a deterministic function of x_{it}. Given the resulting value function EV^k, the probability that household i chooses action d in period t is given by

$$\mathbb{P}^k(d|x_{it}, \omega_t) = \frac{\exp\left(u^k(j, x_{it}, \omega_t) + \beta EV^k(x_{it+1}(j, x_{it}), \omega_{t+1})\right)}{\sum_{j'} \exp\left(u^k(j', x_{it}, \omega_t) + \beta EV^k(x_{it+1}(j', x_{it}), \omega_{t+1})\right)}$$

The transition probabilities between individual states are hence given by

$$\mathbb{P}^k(j', \tau'|j, \tau, \omega) = \begin{cases} \mathbb{P}^k(d = j|j, \tau, \omega) + \mathbb{P}^k(d = s|j, \tau, \omega) & , \tau' = \min(\tau + 1, \overline{\tau}) \text{ and } j = j' \\ \mathbb{P}^k(d = j'|j, \tau, \omega) & , \tau' = 1 \text{ and } j \neq j' \\ 0 & , \text{otherwise} \end{cases}$$

It follows from the above that for any household type k, the one-step transition probabilities are continuous in $(\mathbf{r}, \mathbf{a}) \in \mathcal{D}$, and since the flow utility function and the EV^k functions are both strictly decreasing in $\tilde{r}_j, \forall j \in \mathcal{J}$ such that $\tilde{r}_j \in [\min\{-H_j, \log\left(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\right)\}, \log(\psi_k)]$, it follows that the transition probabilities are strictly decreasing in r_j on this domain.

We stack up the transition probabilities into a transition matrix $\mathbf{\Pi}^k$

$$\mathbf{\Pi}^k(\omega) = \left[\mathbb{P}^k(j', \tau'|j, \tau, \omega)\right]_{j \in \mathcal{J} \cup \{\varnothing\}, \tau \in \{1, \ldots, \overline{\tau}\}}$$

Observe that the resulting Markov chain is regular if

$$\tilde{r}_j \in \left[\min\left\{ -H_j, \log\left(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\right) \right\}, \log(\psi_k) \right], \forall j \in \mathcal{J}$$

To this end, suppose $\tilde{r}_j < \log(\psi_k), \forall j \in \mathcal{J}$. We claim that the entries of $\mathbf{\Pi}^k(\omega)^{\bar{\tau}+2}$ are all strictly positive. Indeed, this follows from (i) $\mathbb{P}^k(d = j'|j, \tau, \omega) > 0, \forall j, \tau, j' \neq j$, (ii) $\mathbb{P}^k(d = s|j, \tau, \omega) > 0, \forall j, \tau$, and (iii) it takes at most two steps to arrive to a state $(j, 1), \forall j \in \mathcal{J}$. Whenever $\exists j \in \mathcal{J}$ such that $\tilde{r}_j > \log(\psi_k)$, since such a location is not affordable, the household will never select it. Hence, in this case we may restrict the Markov chain to the rest of the locations, and the restricted chain is again regular by an analogous argument. It follows a stationary distribution exists, is unique, and equals the limiting distribution. Denote the stationary distribution by $\text{ß}^k(\omega) \equiv \text{ß}^k(\mathbf{\Pi}^k(\omega))$.

Next, we claim $\text{ß}^k(\omega)$ is a continuous function of ω. To this end, we define the following auxiliary matrices following Schweitzer (1968). The time-averaged transition matrix (which always exists for ergodic Markov chains (Bharucha-Reid, 1997)) is given by

$$\mathbf{\Pi}^{k,\infty}(\omega) = \lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} \mathbf{\Pi}^k(\omega)^j$$

The fundamental matrix of Kemeny and Snell (1983) is then given by

$$\mathbf{Z}^k(\omega) = \left(\mathbf{I} - \mathbf{\Pi}^k(\omega) + \mathbf{\Pi}^{k,\infty}(\omega)\right)^{-1}$$

where $\mathbf{I}$ is the identity matrix. For any feasible ω, ω', define $\Delta^k(\omega, \omega') \equiv \mathbf{\Pi}^k(\omega') - \mathbf{\Pi}^k(\omega)$, and vector $\mathbf{g}^k(\omega, \omega')$ given by

$$g_l^k(\omega, \omega') \equiv \sum_{s,m=1}^{N} \pi_s^k(\mathbf{\Pi}^k(\omega)) Z_{sl}^k(\omega) \Delta_{sl}^k(\omega, \omega')$$

It follows from expression 15 in Schweitzer (1968)

$$\begin{aligned}
||\text{ß}^k(\omega) - \text{ß}^k(\omega')||_\infty &= ||\text{ß}^k(\mathbf{\Pi}^k(\omega)) - \text{ß}^k(\mathbf{\Pi}^k(\omega'))||_\infty \\
&= ||\text{ß}^k(\mathbf{\Pi}^k(\omega)) - \text{ß}^k(\mathbf{\Pi}^k(\omega) + \Delta^k(\omega,\omega'))||_\infty \\
&= ||\text{ß}^k(\mathbf{\Pi}^k(\omega)) - \big(\text{ß}^k(\mathbf{\Pi}^k(\omega)) + \mathbf{g}^k(\omega,\omega') + \mathcal{O}\big(\Delta^k(\omega,\omega')^2\big)\big)||_\infty \\
&\leqslant ||\mathbf{g}^k(\omega,\omega')||_\infty + ||\mathcal{O}\big(\Delta^k(\omega,\omega')^2\big)||_\infty
\end{aligned}$$

Further, we have

$$||\mathbf{g}^k(\omega,\omega')||_\infty \leqslant ||\mathbf{Z}^k(\omega)||_\infty ||\Delta^k(\omega,\omega')||_\infty \to \mathbf{0} \text{ as } \omega' \to \omega$$

since $\mathbf{\Pi}^k(\omega)$ is a continuous function of ω and so $||\Delta^k(\omega,\omega')||_\infty \to \mathbf{0}$ as $\omega' \to \omega$. Similarly, $||\mathcal{O}\big(\Delta^k(\omega,\omega')^2\big)||_\infty \to \mathbf{0}$ as $\omega' \to \omega$. Hence, $||\text{ß}^k(\omega) - \text{ß}^k(\omega')||_\infty \to \mathbf{0}$ as $\omega' \to \omega$. Since ω was arbitrary, it follows that $\pi^k(\omega)$ is a continuous function of ω.

We also claim that $\sum_\tau \pi^k_{j\tau}(\omega)$ is a decreasing function of $\tilde{r}_j$ and an increasing function of $\tilde{r}_{j'}$ for $j \neq j'$. Indeed, this relation follows directly from the definition of a stationary distribution

$$\sum_\tau \pi^k_{j\tau}(\omega) = \sum_\tau \sum_{j',\tau'} \mathbf{\Pi}^k_{(j,\tau),(j',\tau')}(\omega)\pi^k_{j',\tau'}(\omega)$$

since $\mathbf{\Pi}^k_{(j,\tau),(j',\tau')}(\omega)$ is a decreasing function of $\tilde{r}_j$, $\forall \tau, j', \tau'$, and weakly increasing of $\tilde{r}_l$ for $l \neq j$, and since probabilities sum to 1.

As in the model description, we define the demand function for squared footage of long-term housing by

$$\mathcal{Q}^{D,L,k}_j(\omega) = \frac{\alpha^k_h w^k M_k \sum_\tau \pi^k_{j\tau}(\omega)}{e^{\tilde{r}_j}}$$

From the above, it follows that $\mathcal{Q}^{D,L,k}_j(\omega)$ is a continuous function of ω and that $\mathcal{Q}^{D,L,k}_j(\omega)$ is decreasing in $\tilde{r}_j$ and weakly increasing in $\tilde{r}_{j'}$ for $j \neq j'$. The aggregate demand function $\mathcal{Q}^{D,L}_j(\omega) = \sum_k \mathcal{Q}^{D,L,k}_j(\omega)$ inherits these properties. Further, we correspondingly redefine the share of long-term houses as

$$s_j^L(\tilde{\mathbf{r}}) = \frac{\exp(\alpha e^{\tilde{r}_j})}{\exp(\alpha e^{\tilde{r}_j}) + \exp(\alpha p_j + \kappa_j)}$$

Fixing amenities $\mathbf{a}$ and log-rent $\tilde{\mathbf{r}}$, solving for equilibrium in the market for short-term housing corresponds to solving for a root of $\mathbf{z}^S(\tilde{\mathbf{r}}, \mathbf{p}, \mathbf{a})$ in $\mathbf{p}$. Since Airbnb tourists can choose only among inner-city neighborhoods, the economy satisfies standard conditions for the existence of competitive equilibria (see, e.g., Mas-Colell, Whinston, Green et al. (1995)). Equilibrium exists by an application of Kakutani's fixed-point theorem on the unit simplex of normalized price vectors (Mas-Colell et al., 1995). Moreover, the gross substitute property also holds by standard arguments. Therefore, the equilibrium is unique. Hence, we can define the equilibrium price vector as a function of log-rent and amenities $\mathbf{p}^*(\tilde{\mathbf{r}}, \mathbf{a})$. The continuity of this function follows by applying the Implicit Function Theorem to $\mathbf{z}^S$ at market clearing prices.

Solving for equilibrium in the market for long-term housing corresponds to solving the following system of equations in $\tilde{\mathbf{r}} \in \mathbb{R}^J$

$$\mathcal{Q}_j^{D,L}(\tilde{\mathbf{r}}, \mathbf{a}) = s_j^L(\tilde{\mathbf{r}}, \mathbf{p}^*(\tilde{\mathbf{r}}, \mathbf{a}))H_j, \forall j \in \mathcal{J}$$

We claim this system has a solution for fixed amenities $\mathbf{a} \in [0, \sum_{k=1}^K \frac{M_k \alpha_s^k \alpha_c^k}{F_{js}\sigma_s}]^{J \cdot S}, j \in \mathcal{J}$. Before we proceed, note that the following holds using the above and by Assumption 1. With a slight abuse of notation, if we denote by

$$\tilde{r}_{min} \equiv \log\left(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\right)$$

we have

$$\lim_{\tilde{r}_j^L \to \tilde{r}_{min}} z_j(\tilde{\mathbf{r}}) = \frac{\sum_{k=1}^K \alpha_h^k w^k M_k \sum_\tau \pi_{j\tau}^k(\tilde{r}_{min}, \tilde{\mathbf{r}}_{-j})}{q \cdot \min_k \alpha_h^k w^k} \sum_{j=1}^J H_j - s_j^L(\tilde{r}_{min}, \tilde{\mathbf{r}}_{-j}) H_j$$

$$\geqslant \frac{\sum_{k=1}^K M_k \sum_\tau \pi_{j\tau}^k(\tilde{r}_{min}, \tilde{\mathbf{r}}_{-j})}{q} \sum_{j=1}^J H_j - s_j^L(\tilde{r}_{min}, \tilde{\mathbf{r}}_{-j}) H_j$$

$$\geqslant \sum_{k=1}^K M_k \times \sum_{j=1}^J H_j - s_j^L(\tilde{r}_{min}, \tilde{\mathbf{r}}_{-j}) H_j > 0$$

and

$$\lim_{\tilde{r}_j \to \log(\psi_{max})} z_j^L(\tilde{\mathbf{r}}) = -s_j^L(\log(\psi_{max}), \tilde{\mathbf{r}}_{-j}) H_j < 0$$

We transform the root-finding problem to a fixed-point problem by defining $\mathbf{f} : \mathbb{R}^J \to \mathbb{R}^J$, $\mathbf{f}(\tilde{\mathbf{r}}) = \mathbf{z}(\tilde{\mathbf{r}}) + \tilde{\mathbf{r}}$. Observe that $\mathbf{f}$ is continuous, and since z_j is decreasing in $\tilde{r}_j$ and by the above, we must have

$$\mathbf{f}\Big(\bigtimes_{j \in \mathcal{J}} \left[\min\left\{ -H_j, \log\Big(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\Big) \right\}, \log\left(\max\Big\{ \sum_k \frac{\alpha_h^k w^k M_k}{\psi_{min}}, \psi_{max} \Big\} \right) \right] \Big) \subseteq$$

$$\bigtimes_{j \in \mathcal{J}} \left[\min\left\{ -H_j, \log\Big(\frac{q \cdot \min_k \alpha_h^k w_k}{\sum_{j=1}^J H_j}\Big) \right\}, \log\left(\max\Big\{ \sum_k \frac{\alpha_h^k w^k M_k}{\psi_{min}}, \psi_{max} \Big\} \right) \right]$$

where the considered set is convex and compact. Applying Brouwer's fixed point theorem, an equilibrium exists. Finally, an equilibrium vector of strictly positive rental prices must exist by the properties of the logarithmic function.

Further, since the aggregate demand function is strictly decreasing in $r_j, \forall j \in \mathcal{J}$ such that $r_j \leqslant \psi_{\max}$ and since all equilibrium prices are at most $\psi_{\max}$, restricting attention to $(0, \psi_{max}]$, the strict gross substitutes property holds in this case, as well. Hence, the equilibrium is unique. This allows us to define the equilibrium price vector as a function of the fixed vector of amenities $\mathbf{r}^*(\mathbf{a})$. Continuity of this function follows by applying the Implicit Function Theorem to $\mathbf{z}^L$ at market-clearing prices.

Equilibrium in the market for amenities requires, $\forall j \in \mathcal{J}, s \in \mathcal{S}$,

$$a_{js} = \frac{1}{F_{js}\sigma_s} \left(\sum_{k=1}^{K} \mathcal{Q}_j^{D,L,k}(\mathbf{r}^*(\mathbf{a}), \mathbf{a}) \alpha_s^k \alpha_c^k w^k + \mathcal{Q}_j^T(\mathbf{p}^*(\mathbf{r}^*(\mathbf{a}), \mathbf{a}), \mathbf{a}) \alpha_s^T \alpha_c^T w^T \right)$$

Define $\psi : \mathbb{R}^{J \cdot S} \to \mathbb{R}^{J \cdot S}$ by, $\forall j \in \mathcal{J}, s \in \mathcal{S}$,

$$\psi_{js}(\mathbf{a}) = \frac{1}{F_{js}\sigma_s} \left(\sum_{k=1}^{K} \mathcal{Q}_j^{D,L,k}(\mathbf{r}^*(\mathbf{a}), \mathbf{a}) \alpha_s^k \alpha_c^k w^k + \mathcal{Q}_j^T(\mathbf{p}^*(\mathbf{r}^*(\mathbf{a}), \mathbf{a}), \mathbf{a}) \alpha_s^T \alpha_c^T w^T \right).$$

By the above, it follows that ψ is continuous and

$$\psi\left(\left[0, \frac{1}{F_{js}\sigma_s} \Big(\sum_{k=1}^{K} M^k \alpha_s^k \alpha_c^k w^k + M^T \alpha_s^T \alpha_c^T w^T \Big) \right]^{J \cdot S} \right) \\ \subseteq \left[0, \frac{1}{F_{js}\sigma_s} \Big(\sum_{k=1}^{K} M^k \alpha_s^k \alpha_c^k w^k + M^T \alpha_s^T \alpha_c^T w^T \Big) \right]^{J \cdot S}$$

Existence of equilibrium hence follows by Brouwer's fixed-point theorem. ■

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